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Shumacher appends a solution handed to him by Gauss (*Carl Friedrich Gauss Werke*, vol. 4, 2ter Abdruck, 1880, pp. 406–407); this leads to the equation

$$A^2 - (\alpha + \beta + \gamma + \delta + \epsilon)A + (\alpha\beta + \beta\gamma + \gamma\delta + \delta\epsilon + \epsilon\alpha) = 0.$$

P. Serret noted (*Nouvelles Annales de Mathêmatiques*, vol. 7, 1848, p. 28) that if the pentagon becomes the quadrilateral $ABCD$, α, β, γ being the areas of the triangles ABC, BCD, CDA, the above equation for the area of the quadrilateral becomes $A^2 - (\alpha + \beta + \gamma)A + \alpha\beta + \beta\gamma = 0$,—which can, on solving, be at once verified.

The problem of the construction with ruler and compasses of an inscribed quadrilateral, being given its sides, has an interesting history extending over five hundred years. This is, in general, only a particular case of the problem discussed by Ozanam (*Dictionaire Mathematique*, Amsterdam, 1691, pages 461–464): "Construct the quadrilateral, of given area, given the lengths of its four sides." R. C. ARCHIBALD.

SOLUTIONS

2832 [1920, 227]. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that the square of the sum of four squares is the sum of four squares, that the square of the sum of eight squares is the sum of six squares, and that the square of the sum of sixteen squares is the sum of ten squares.

I. SOLUTION BY THE PROPOSER.

Given the identity,

$$\begin{aligned}
&(Pp+Pq+Qp-Qq-Rr+Rs-Sr-Ss-Tt-Tu+Ut-Uu-Vv-Vx+Xv-Xx)^2\\
&\quad+(Pr+Ps-Qr+Qs-Rp+Rq+Sp+Sq-Tv+Tx+Uv+Ux+Vt-Vu-Xt-Xu)^2\\
&\quad+(Pp-Pq-Qp-Qq+Rr+Rs-Sr+Ss+Tt-Tu+Ut+Uu+Vv-Vx+Xv+Xx)^2\\
&\quad+(Pr-Ps+Qr+Qs+Rp+Rq+Sp-Sq+Tv+Tx+Uv-Ux-Vt-Vu-Xt+Xu)^2\\
&\quad+(Pt+Pu+Qt-Qu-Rv-Rx+Sv-Sx+Tp+Tq-Up+Uq+Vr-Vs+Xr+Xs)^2\\
&\quad+(Pv-Px-Qv-Qx-Rt+Ru-St-Su+Tr+Ts-Ur+Us-Vp+Vq-Xp-Xq)^2\\
&\quad+(Pt-Pu-Qt-Qu+Rv-Rx+Sv+Sx-Tp+Tq-Up-Uq-Vr-Vs+Xr-Xs)^2\\
&\quad+(Pv+Px+Qv-Qx+Rt+Ru-St+Su-Tr+Ts-Ur-Us+Vp+Vq-Xp+Xq)^2\\
&\qquad\equiv 2(P^2+Q^2+R^2+S^2+T^2+U^2+V^2+X^2)(p^2+q^2+r^2+s^2+t^2+u^2+v^2+x^2). \qquad (1)
\end{aligned}$$

Add $(P^2 + Q^2 + R^2 + S^2 + T^2 + U^2 + V^2 + X^2)^2$ and $(p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + x^2)^2$ to each member of (1). The second member then becomes the square of the sum of 16 squares, and the first member becomes the sum of 10 squares, as required. If $P = Q = R = S = t = u = v = x =$ zero, it follows that the square of the sum of eight squares is the sum of six squares. That the square of the sum of four squares is the sum of four squares is a direct consequence of Euler's well known theorem: (sum of four squares) (sum of four squares) = (sum of four squares).

II. Mr. NORMAN ANNING, of the University of Michigan, contributes formulæ exhibiting these squares as the sum of 3, 5 and 9 squares, instead of 4, 6, and 10. The results are given for record.

Corresponding to $(x_1^2 + x_2^2)^2 = (x_1^2 - x_2^2)^2 + 4x_1^2x_2^2$, we have

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 = (x_1^2 - x_2^2 + x_3^2 - x_4^2)^2 + 4(x_1^2 + x_3^2)(x_2^2 + x_4^2)$$

$$= \text{the sum of three squares.}$$

$$(x_1^2 + x_2^2 + \cdots + x_8^2)^2$$

$$= (x_1^2 - x_2^2 + - + \cdots - x_8^2)^2 + 4(x_1^2 + x_3^2 + x_5^2 + x_7^2)(x_2^2 + x_4^2 + x_6^2 + x_8^2)$$

= the sum of five squares, since by Euler's Identity the last term can be expressed as the sum of four squares.

Similarly,

$$(x_1^2 + x_2^2 + \cdots + x_{16}^2)^2$$

$$= (x_1^2 - x_2^2 + - + \cdots - x_{16}^2)^2 + 4(x_1^2 + x_3^2 + \cdots + x_{15}^2)(x_2^2 + x_4^2 + \cdots + x_{16}^2)$$

= the sum of nine squares.

The fact that the product of two numbers each of which is the sum of eight squares is itself the sum of eight squares is stated, with a reference to "Thomson, 1877," on page 115 of Carmichael's *Diophantine Analysis*, 1915.

An identity for the eight-square case is as follows:

$$\begin{aligned}
&(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) \times (p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + w^2)\\
&\quad \equiv (ap + bq + cr + ds + et + fu + gv + hw)^2 + (aq - bp + cv - du + ew + fs - gr - ht)^2\\
&\quad + (ar - bv - cp - dt + es - fw + gq + hu)^2 + (as + bu + ct - dp - er - fq - gw + hv)^2\\
&\quad + (at - bw - cs + dr - ep + fv - gu + hq)^2 + (au - bs + cw + dq - ev - fp + gt - hr)^2\\
&\quad + (av + br - cq + dw + eu - ft - gp - hs)^2 + (aw + bt - cu - dv - eq + fr + gs - hp)^2.
\end{aligned}$$

NOTE (written after the above was in type)—Euler's identity given in a letter to Goldbach dated May 4, 1748 (*Correspondance Mathématique et Physique* ed. by Fuss, vol. 1, 1843, p. 452) was as follows: "Si $m = aa + bb + cc + dd$ et $n = pp + qq + rr + ss$ erit $mn = A^2 + B^2 + C^2 + D^2$ existente $A = ap + bq + cr + ds$, $B = aq - bp - cs + dr$, $C = ar + bs - cp - dq$, $D = as - br + cq - dp$." This was generalized by C. F. Degen in 1822 (*Mém. Acad. Sc. St. Pétersbourg*, vol. 8 [1817–18], pp. 207–219) as follows:

$$\begin{aligned}
&(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) \times (p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + w^2)\\
&\quad \equiv (ap + bq + cr + ds + et + fu + gv + hw)^2 + (aq - bp + cs - dr + eu - ft + gw - hv)^2\\
&\quad + (ar - bs - cp + dq \mp ev \pm fw \pm gt \mp hu)^2 + (as + br - cq - dp \pm ew \pm fv \mp gu \mp ht)^2\\
&\quad + (at - bu \pm cv \mp dw - ep + fq \mp gr \pm hs)^2 + (au + bt \mp cw \mp dv - eq - fp \pm gs \pm hr)^2\\
&\quad + (av - bw \mp ct \pm du \pm er \mp fs - gp + hq)^2 + (aw + bv \pm cu \pm dt \mp es \mp fr - gq - hp)^2.
\end{aligned}$$

Cf. L. E. Dickson, "On quaternions and their generalization and the history of the eight square theorem," *Annals of Mathematics*, second series, volume 20, p. 164, 1919.

2840 [1920, 274]. Proposed by NORMAN ANNING, University of Michigan.

It is observed in a table of values of

$$\log_{10}(\operatorname{colog}_{10} x)$$

that second differences are zero for values of x in the neighborhood of 0.37. Prove that this must be the case. (Cf. Chappell's *Five-Figure Mathematical Tables*, Edinburgh, 1915, p. 180.)

SOLUTION BY C. C. WYLIE, University of Illinois.

Let $y = \log_{10}(\operatorname{colog}_{10} x)$ and $M = \log_{10} e$. Then

$$y = M \log\left(\log\frac{1}{x}\right) + M \log M.$$

$$\frac{dy}{dx} = -\frac{M}{x\log\frac{1}{x}}, \quad \frac{d^2y}{dx^2} = \frac{M\left(\log\frac{1}{x} - 1\right)}{\left(x\log\frac{1}{x}\right)^2} = 0,$$